Coach Monks's MathCounts Playbook!

Secret facts that every Mathlete should know in order to win!

Learn the items marked with a ⋆ first. Then once you have mastered them try to learn the other topics.

1. ⋆ **Squares and Cubes of small integers, powers of 2, decimals represenation of common fractions**

n	n^2	n^3	2^n	$n!$
1	1	1	2	1
2	4	8	4	2
3	9	27	8	6
4	16	64	16	24
5	25	125	32	120
6	36	216	64	720
7	49	343	128	5040
8	64	512	256	
9	81	729	512	
10	100	1000	1024	

n	n^2
11	121
12	144
13	169
14	196
15	225
16	256
17	289
18	324
19	361
20	400

fraction	decimal	fraction	decimal
1/2	0.5	5/6	$0.8\overline{3}$
1/3	$0.\overline{3}$	1/7	$0.\overline{142857}$
2/3	$0.\overline{6}$	1/8	0.125
1/4	0.25	3/8	0.375
3/4	0.75	5/8	0.625
1/5	0.2	7/8	0.875
2/5	0.4	1/9	$0.\overline{1}$
3/5	0.6	1/10	0.1
4/5	0.8	1/12	$0.08\overline{3}$
1/6	$0.1\overline{6}$	1/20	0.05

2. Divisibility

a. ⋆ **Prime Factorization** - a divides b if and only if the exponents of the prime factors of a are less than or equal to the exponents of the corresponing prime factors of b

b. ⋆ **Primality Testing -** to test a positive integer a to see if its prime, we only need to check to see if it is divisible by the primes p such that $p^2 \leq a$.

c. ⋆ **Number of divisors** - to find the number of positive divisors of a positive whole number n :

1. Write n as a product of prime powers
2. Add one to each of the exponents of the primes
3. Multiply these new exponents together

i.e. if $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ is the prime factorization of n, then n has $(e_1+1)(e_2+1)\cdots(e_k+1)$ positive divisors.

d. **Sum of all divisors:** To find the sum of all of the divisors of a positive integer:

1. Write n as a product of prime powers
2. Compute the sum of all powers of each prime (from zero up to exponent inclusive)
3. Multiply these sums together

i.e. if $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ is the prime factorization of n, then the sum of the divisors of n is $(1+p_1+p_1^2+\cdots+p_1^{e_1})(1+p_2+p_2^2+\cdots+p_2^{e_2})\cdots(1+p_k+p_k^2+\cdots+p_k^{e_k})$. Note that these factors can be computed using the geometric series formula from #5c below, so the sum of the divisors of n is also

$$\left(\frac{p_1^{e_1+1}-1}{p_1-1}\right)\left(\frac{p_2^{e_2+1}-1}{p_2-1}\right)\cdots\left(\frac{p_k^{e_k+1}-1}{p_k-1}\right)$$

e. ⋆ **Divisibility Tests**

n is divisible by:	if:
2	the rightmost digit of n is $0, 2, 4, 6,$ or 8
3	the sum of the digits of n is divisible by 3
4	the number formed by erasing all but the rightmost two digits of n is divisible by 4
5	the rightmost digit of n is 0 or 5
6	n is divisible by 2 and also by 3
7	shift each digit two columns to the right and double it, and then add it to the existing digit mod 7
8	the number formed by erasing all but the rightmost three digits of n is divisible by 8
9	the sum of the digits of n is divisible by 9
10	the rightmost digit of n is 0
11	the difference between the sum of every other digit of n and the sum of the remaining digits is divisible by 11

3. GCD and LCM

a. to get the prime factorization of $\gcd(a, b)$ write down the prime factorizations of a, b and take the least power of each prime that appears in both

b. to get the prime factorization of $\mathrm{lcm}(a, b)$ write down the prime factorizations of a, b and take the largest power of each prime that appears in either

c. $\mathrm{lcm}(a, b)\gcd(a, b) = ab$

d. If $\gcd(a, b) = 1$ then $ab - a - b$ is the largest integer which cannot be written in the form $as + bt$ for some integers $s \geq 0$, $t \geq 0$.

4. Modular Arithmetic

a. Definition: x mod n is the remainder when x is divided by n

b. Modular arithmetic: If $a = x \bmod n$ and $b = y \bmod n$ then

$$(x + y) \bmod n = (a + b) \bmod n$$

$$(xy) \bmod n = (ab) \bmod n$$

c. All of the divisibility tests above can also be used to compute the remainder (except for 6). For $n \bmod 11$, remember that $(1 \bmod 11) = 1$ and $(10 \bmod 11) = -1 \bmod 11$ (so you know which digit sum to subtract). To compute $n \bmod 6$, add the rightmost digit to 4 times the sum of the remaining digits and then compute the total mod 6.

5. Sequences

Name	Description	Sequence	n^{th} term
Periodic	repeats a finite sequence over and over	$\overline{a_0, a_1, \ldots, a_{k-1}}$	$a_{(n-1) \bmod k}$
Arithmetic	difference between two consecutive terms is d	$a, a+d, a+2d, a+3d, \ldots$	$a+(n-1)d$
Geometric	ratio of two consecutive terms is r	$a, ar, ar^2, ar^3, ar^4 \ldots$	ar^{n-1}

6. Adding

a. ⋆ **Gauss's Formula**

$$1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

b. ⋆ **Sum of an Arithmetic Sequence** - is the average of the first and last terms, multiplied by the number of terms (the number of terms is one more than the difference between the first and last terms divided by the common difference)

c. ⋆ **Sum of a Geometric Sequence**

$$1+r+r^2+r^3+\cdots+r^n = \frac{r^{n+1}-1}{r-1}$$

d. An interesting sum

$$n^2-(n-1)^2+\cdots+4^2-3^2+2^2-1^2 = \frac{n(n+1)}{2}$$

(Note if n is odd then the sum is written $n^2-(n-1)^2+\cdots+5^2-4^2+3^2-2^2+1^2 = \frac{n(n+1)}{2}$)

7. Fantastic Fractions

a. Egyption fractions: are fractions whose numerator is 1. Every positive fraction can be written as a finite sum of distinct Egyption fractions by the "greedy" algorithm:

$$\frac{a}{b} = \frac{1}{q+1} + \frac{a-r}{b(q+1)}$$

where a, b are relatively prime postive integers, and q, r are the quotient and remainder when b is divided by a.

b. Two interesting Egyptian fraction sums

$$\frac{1}{n} = \frac{1}{n+1} + \frac{1}{n(n+1)}$$

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n \cdot (n+1)} = \frac{n}{n+1}$$

8. Counting

a. Permutations

i. ⋆ the number of ordered arrangements of n distinct things

$$n!$$

ii. the number of ordered arrangements of k distinct things selected from n distinct things

$$nPk = \frac{n!}{(n-k)!}$$

iii. the number of ordered arrangements of k things selected from n distinct things if duplicates are allowed

$$k^n$$

b. Combinations

i. ⋆ the number of ways to choose a set of k distinct things from a set of n distinct things ("n choose k")

$$nCk = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

ii. the number of ways to choose k things from a set of n things if duplicates are allowed

$$\binom{n+k-1}{k} = \frac{(n+k-1)!}{k!(n-1)!}$$

iii. Pascal's triangle (computes binomial coefficients)

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}$$

iv. Pascal Facts: The rows of Pascal's triangle add up to the powers of 2. The decimal representation of powers of 11 can be read from the first five rows.

v. Counting Paths: To find the number of paths from a starting location to a finish location (on a directed graph) work backwards from the finish and label each node with the number of paths from that node to the finish.

9. Algebra

a. Famous identities

⋆ Difference of two squares:	$x^2 - y^2 = (x-y)(x+y)$
Difference of two cubes:	$x^3 - y^3 = (x-y)(x^2+xy+y^2)$
Sum of two cubes:	$x^3 + y^3 = (x+y)(x^2-xy+y^2)$

b. An Application: Fast arithmetic by difference of two squares

Ex: $97 \cdot 103 = (100-3)(100+3) = 100^2 - 3^2 = 9991$

Ex: $47^2 - 3^2 = (47-3)(47+3) = 44 \cdot 50 = 44 \cdot 100 \cdot \frac{1}{2} = 4400 \cdot \frac{1}{2} = 2200$

Ex: $97^2 = (97+3)(97-3) + 3^2 = 100 \cdot 94 + 9 = 949$

c. Binomial Theorem

$$(x+y)^0 = 1$$
$$(x+y)^1 = x+y$$
$$(x+y)^2 = x^2 + 2xy + y^2$$
$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y$$
$$\vdots$$
$$(x+y)^n = x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \binom{n}{3}x^{n-3}y^3 + \cdots + \binom{n}{n-2}x^2y^{n-2} + \binom{n}{n-1}xy^{n-1} + y^n$$

i.e. the coefficients of the expanded form of $(x+y)^n$ are just a row in Pascal's triangle.

d. **Max/Min of a quadratic:** If a, b, c are real numbers with $a \neq 0$ then the maximum/minimum value of $ax^2 + bx + c$ occurs when $x = -\frac{b}{2a}$.

e. **Symmetry-Product Principle**: As the distance between two positive numbers decreases their product increases if their sum remains constant.

f. **Roots of a quadratic:** the sum of the roots of $ax^2 + bx + c = 0$ is $\frac{-b}{a}$. The product of the roots is $\frac{c}{a}$.

10. ⋆ **Probability**

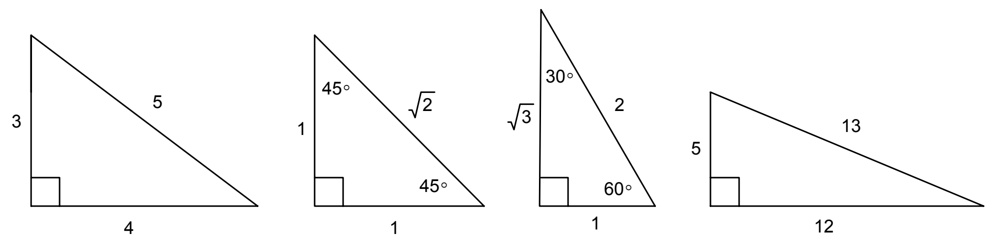
Let S be a set (called the *sample space*) and E a subset of S (called an *event*). Then the probability of obtaining an element of E when selecting an element of S at random is

$$P(E) = \frac{\#(E)}{\#(S)}$$

where $\#(X)$ is the number of elements in the set X, i.e. to compute the probability of something happening you divide the number of ways it can happen by the total number of possible outcomes.

11. Geometry

a. ⋆ Famous triangles (and multiples of these)

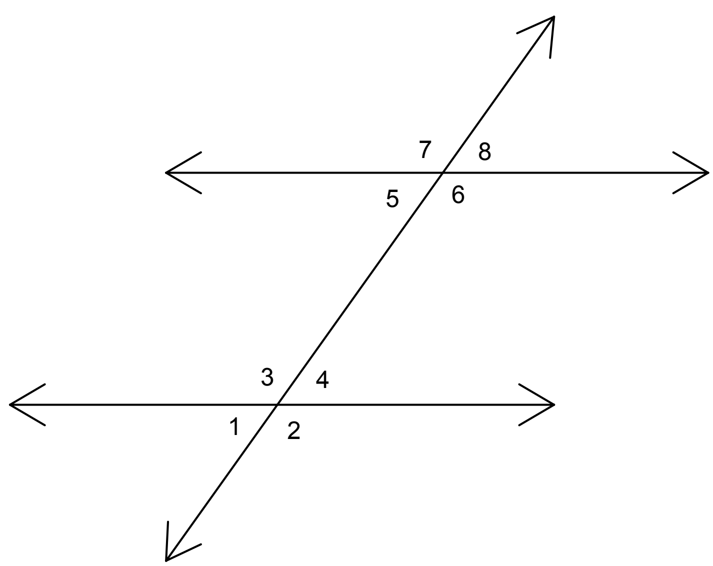


b. Other Pythagorean triples

3,4,5	7,24,25	9,40,41
5,12,13	8,15,17	20,21,29

c. Diagonal of the unit cube: $\sqrt{3}$

d. Names of angle pairs



i. Vertical angles: $\{1,4\}, \{2,3\}, \{5,8\}, \{7,6\}$

ii. Alternate interior angles: $\{4,5\},\{3,6\}$

iii. Alternate exterior angles: $\{2,7\},\{1,8\}$

iv. Same side interior angle : $\{3,5\},\{4,6\}$

v. Same side exterior angles: $\{1,7\},\{2,8\}$

vi. Corresponding angles: $\{1,5\},\{3,7\},\{2,6\},\{4,8\}$

vii. Adjacent angles: $\{1,3\},\{3,4\},\{4,2\},\{1,2\},\{5,7\},\{7,8\},\{8,6\},\{5,6\}$

e. In the figure in part d, lines m and n are parallel if and only if angles $1,4,5,$ and 8 are all congruent to each other (and angles 2, 3, 6, and 7 are congruent to each other)

f. The ratio of the areas of two similar figures is the ratio of the side lengths squared. Similarly, the ratio of the volumes of two similar figures is the ratio of the side lengths cubed.

g. ⋆ Famous volumes, areas, and perimeters

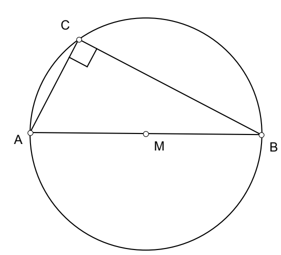
CIRCUMFERENCE	
Circle	$c = \pi d$
	$c = 2\pi r$
AREA	
Square	$A = s^2$
Rectangle	$A = bh$
Parallelogram	$A = bh$
Trapezoid	$A = \frac{(b_1+b_2)}{2}h$
Circle	$A = \pi r^2$
Equilateral Triangle	$A = \frac{\sqrt{3}}{4}s^2$
Triangle	$A = \frac{1}{2}bh$
Triangle (Heron's formula)	$A = \sqrt{s(s-a)(s-b)(s-c)}$ where $s = (a+b+c)/2$
Triangle	$A = rs$ where $s = (a+b+c)/2$ and r =radius of inscribed circle

SURFACE AREA & VOLUME	
Sphere	$SA = 4\pi r^2$
Sphere	$V = \frac{4}{3}\pi r^3$
Rectangular Prism	$V = A_{base}h$
Cylinder	$V = \pi r^2 h$
Cone	$V = \frac{1}{3}\pi r^2 h$
Pyramid	$V = \frac{1}{3}A_{base}h$

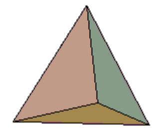
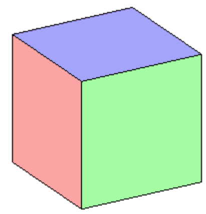
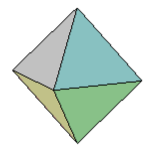
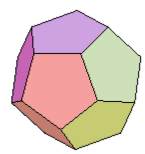
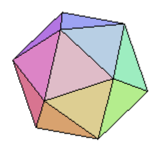
PYTHAGOREAN THEOREM	
Right triangle	$a^2 + b^2 = c^2$
	$c = \sqrt{a^2+b^2}$
	$a = \sqrt{c^2-b^2}$

12. Triangles with Circles:

a. If AB is a diameter of a circle and D is any other point on the circle, then triangle ABC is a right triangle (with hypotenuse AB). Conversely the midpoint of the hypotenuse of a right triangle is equidistant from the three vertices.



13. Regular polyhedra (Platonic Solids):

Name	# Faces	# Edges	# Vertices	Picture
Tetrahedron	4	6	4	
Cube	6	12	8	
Octahedron	8	12	6	
Dodecahedron	12	30	20	
Icosahedron	20	30	12	

14. MathCounts Strategy Tips

a. GUESS!!! There is no penalty for guessing, so be sure to guess at answers that you don't know, especially if there are only a few possible answers.

b. Don't get bogged down. If a problem seems like it will take a long time to complete, just skip it and find problems you can easily answer. Then go back to it if you have time at the end.

c. Look for shorcuts to problems that have an obvious but lengthy solution. For example, $43(17)+57(17)$ can be computed directly, but it is much quicker to use the distributive law: $43(17)+57(17)=(43+57)17=100(17)=1700$. Almost all MathCounts problems have both a short way and a long way to answer them.

d. Always keep your numbers as small as possible when doing calculations by hand. For example, if you have to compute

$$\frac{20 \cdot 35}{15 \cdot 14}$$

don't multiply the top and bottom and then try to reduce the resulting fraction like this:

$$\frac{21 \cdot 35}{15 \cdot 14} = \frac{735}{210} = \textit{YUCK!!}$$

instead divide out all the common factors first:

$$\frac{21 \cdot 35}{15 \cdot 14} = \frac{(3 \cdot 7) \cdot (5 \cdot 7)}{(3 \cdot 5) \cdot (2 \cdot 7)} = \frac{7}{2}.$$

e. Don't spend a lot of time looking for shortcuts to problems if the obvious solution can be done in under 30 seconds!

f. Almost every MathCounts question has a tiny hint about its answer. They almost always specify the form of the answer *only* when that form is not an integer. For example, if they do not say that the answer should be written as a mixed number, improper fraction, or decimal, then its almost certain that the answer is an integer (otherwise there could be confusion in grading the problem). Similarly if they tell you to write the number as a mixed number, it is unlikely that the answer will be an integer. If they tell you to write the answer in simplest radical form, or leave your answer in terms of π, then its a good hint that the answer might contain a radical or π. This can be a big help in detecting careless errors like giving the correct answer in feet when the problem asks for inches.

g. Have FUN!!!